Thermal Diffusivity

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Second Year Laboratory Report

Nov 2004

Abstract

We investigated the thermal diffusivity of a cylindrical block of resin by observing the temperature as a function of time at it's centre with respect to an outside change of temperature. This was done in two different ways; a step method (involving a single change from ice water to boiling water) and a periodic method (involving multiple changes from ice water to boiling water). The result was $(9.45 \pm 0.22)\times 10^{-8}\, m^2 s^{-1}$.

1. Introduction

In an electrical conductor, the resistance of the conductor, amongst other things, depends upon the shape of the conductor and the resistivity of the material from which it is made from.

This is akin to the properties displayed by a thermal conductor. However the conductivity of the thermal conductor is analogous to the resistivity of the electrical conductor.

But neither of these situations depend on time. To include time dependence into the situation we must define a new quantity; diffusivity.

The study of thermal diffusivity is useful in a wide range of fields including building science, material science, meteorology, the design of heat flux sensors and has even been linked to the effect dentures have on some people's sense of taste.

2. Theory

2.1.Step Method

The conduction Equation is[1]; $\frac{\partial\theta}{\partial t} = D\nabla^2\theta$

Where θ is the temperature as a function of position and time.

By expanding this in cylindrical polar coordinates and neglecting ϕ and z

(i.e. $\theta = \theta(r,t)$) dependence; $\frac{\partial\theta}{\partial t} = D\left[\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r}\right]$

The solution to this along the axis of the cylinder is a series of exponential terms:

$$\theta(0,t) = \theta_0 + (\theta_1 + \theta_0)\left[1 - \sum_{n=1}^{\infty} a_n e^{\frac{-\lambda_n^2 Dt}{a^2}}\right]$$

Where λ_n are the positive roots of the Bessel function, i.e. $\lambda_1 \approx 2.405$, $\lambda_2 \approx 5.520$, $\lambda_3 \approx 8.653$ etc. By substituting these values it is observed that a close approximation is to neglect the $n > 1$ terms:

$$\theta = \theta_0 + (\theta_1 + \theta_0)\left[1 - a_1 e^{\frac{-5.784Dt}{a^2}} - a_2 e^{\frac{-30.470Dt}{a^2}} - a_3 e^{\frac{-74.87Dt}{a^2}} - \ldots\ldots\ldots\ldots\right]$$

This approximation gives; $\theta = \theta_1 - a_1 e^{\frac{-5.784Dt}{a^2}}$

i.e $\frac{|\theta - \theta_1|}{\Delta\theta} \propto e^{\frac{-\lambda_1^2 Dt}{a^2}}$ [Equation 1]

Where D : the thermal diffusivity of the block
t : time/ s
a : radius of the cylindrical block/ m
$\lambda_1 = 2.405$

Equation 1 can be rearranged to; $|\theta - \theta_1| \propto \Delta\theta e^{\frac{-\lambda_1^2 Dt}{a^2}}$
Hence using natural logarithms[2]:

$$\ln(\theta - \theta_1) = \ln\Delta\theta + \ln e^{\frac{-\lambda_1^2 Dt}{a^2}} \equiv \ln\Delta\theta - \frac{\lambda_1^2 Dt}{a^2} \qquad \text{[Equation 2]}$$

Hence plotting the logarithm of the difference of the current temperature with the external temperature against time will yield a linear plot with gradient m;

$$m = -\frac{\lambda_1^2 D}{a^2} \qquad \text{[Equation 3]}$$

This can be rearranged to find the diffusivity of the block;

$$D = -\frac{ma^2}{\lambda_1^2} \qquad \text{[Equation 4]}$$

However, using vernier callipers gives a measurement of the diameter not the radius, hence substituting for the diameter, $d = 2a$, gives:

$$D = -\frac{md^2}{4\lambda_1^2} \qquad \text{[Equation 5]}$$

2.2. Periodic Method

Using a similar set up to the Step Method the resin is cooled to be 0ºC and then heated to 100ºC. However this time the resin is spends a certain amount of time in the ice and a certain amount of time in the boiling water. This cycle of ice and boiling water is repeated a few times to give a periodic change in the temperature of the resin.

Using this method the diffusivity of the resin can be found.[4]

As the temperature is a function of radius and the periodic change of the external temperature; $\theta(r,t) = f(r)e^{i\omega t}$

Substituting this into the conduction equation; $i\omega f = D\left[\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r}\right]$

Changing the variable to $z = r\sqrt{\frac{-i\omega}{D}}$ gives; $\frac{\partial^2 f}{\partial z^2} + \frac{1}{z}\frac{\partial f}{\partial z} + f = 0$

The solution to this equation is the Kelvin function, $M_0(x)$. As our periodic change of external temperature is actually a square wave (i.e. it is not a sinusoidal change in temperature), $\theta(r,t)$ can be expanded as a Fourier series to give;

$$\frac{\theta(a,t)}{\theta(0,t)} = \frac{M_0\left(a\sqrt{\frac{\omega}{D}}\right)}{M_0(0)} \qquad \text{[Equation 6]}$$

This can be rearranged to give two different equations, one concerning the amplitudes of the axial temperature and the external temperature:

$$M_0(x) = \frac{4}{\pi}\frac{\theta_2 - \theta_1}{B} \qquad \text{[Equation 7a]}$$

And the other concerning the phase difference between the axial temperature and the external temperature;

$$\phi = \frac{2\pi\,\Delta t}{T} = \arg[M_0(x)] \qquad \text{[Equation 7b]}$$

Where; $x = a\sqrt{\frac{2\pi}{TD}}$ [Equation 8]

Notice that x is a unitless quantity.

Where B : The difference between the maxima temperature and the minima temperature
M_0 : The Kelvin function
θ_2 : The temperature of the boiling water
θ_1 : The temperature of the ice water
ϕ : The phase difference between the axial and external temperatures
Δt : The time difference between a peak of the axial and external temperatures

Thus by plotting the Kelvin function, the value of the diffusivity can be found.

3. Method

3.3. Step Method

To find the diffusivity of the sample of resin, it was cooled using ice to the point where it's temperature was a minimum. Inside the resin is a temperature probe which gives a plot of the axial temperature (the temperature at the centre of the resin) with time. At this point the axial temperature was assumed to be 0ºC. The resin was then placed in boiling water and the axial temperature increased until it reached a maxima, this temperature was assumed to be 100 ºC.

3.4. Periodic Method

Using the same setup as for the Step Method, the resin was cooled to 0ºC and was then placed in boiling water. However for the periodic method, the resin was not allowed to reach a maximum axial temperature, but instead the resin was left in the boiling water for a set amount of time. At the end of this time, the resin was once again placed in the ice water for the same amount of time as before. This process was repeated until a few periods of the axial temperature had been completed.

4. Results

4.1. Results from the Step Method

4.1.1. Trial 1

For this trial the resin was cooled to the minimum axial temperature and was then left in the boiling water for 9 minutes to give the following plot;

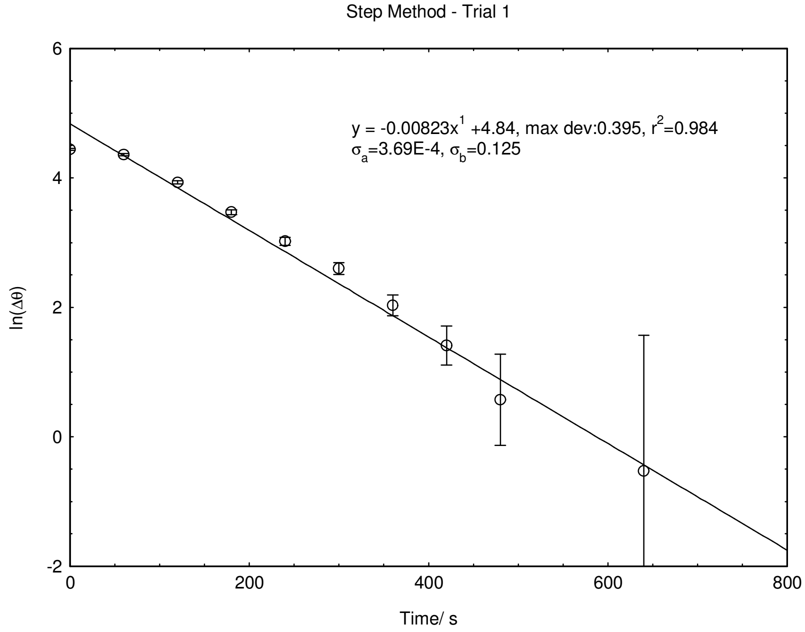

Figure 1 – Results of Trial 1 of the Step Method

Using Equation 5; $D = 12.60\times10^{-8}\,m^2 s^{-1}$

Using the standard methods[3] the value of Chi Squared is found to be; $\chi^2 = 188.1$. This is a high value but can easily be rectified by removing the first point. This point has that smallest error and so will have the greatest effect on reducing the value of Chi Squared.

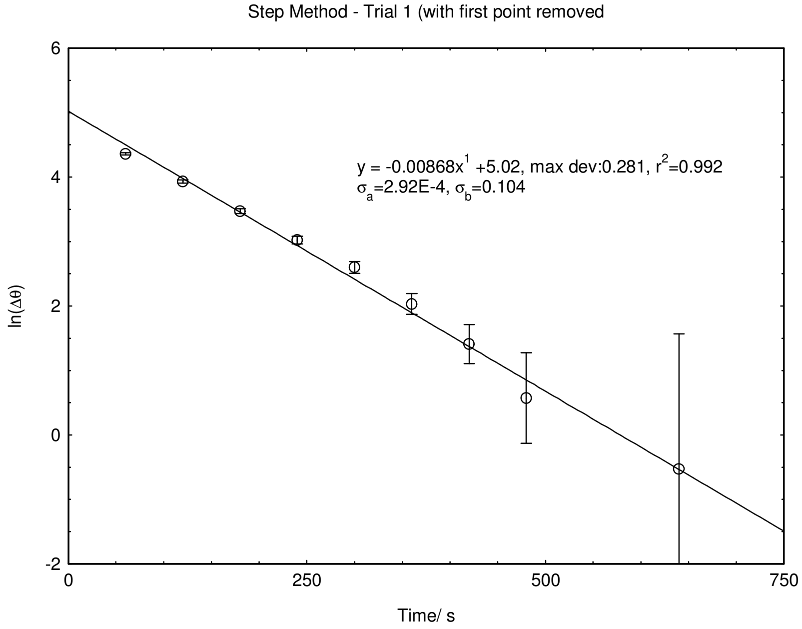

Figure 2 – Results of Trial 1 of the Step Method (with first point removed)

Removing this point, reduces the value of Chi Squared to 2.537, which is far more acceptable. This gives; $m = -0.0087 s^{-1} \Rightarrow D = 13.70\times10^{-8}\,m^2 s^{-1}$. (By Equation 5)

4.1.2. Trial 2

Unlike the first trial, the resin was heated to the maximum axial temperature of 100 ºC in the boiling water and then placed in the ice water to observe how the axial temperature varies as the resin is cooled for 22 minutes to give the following plot;

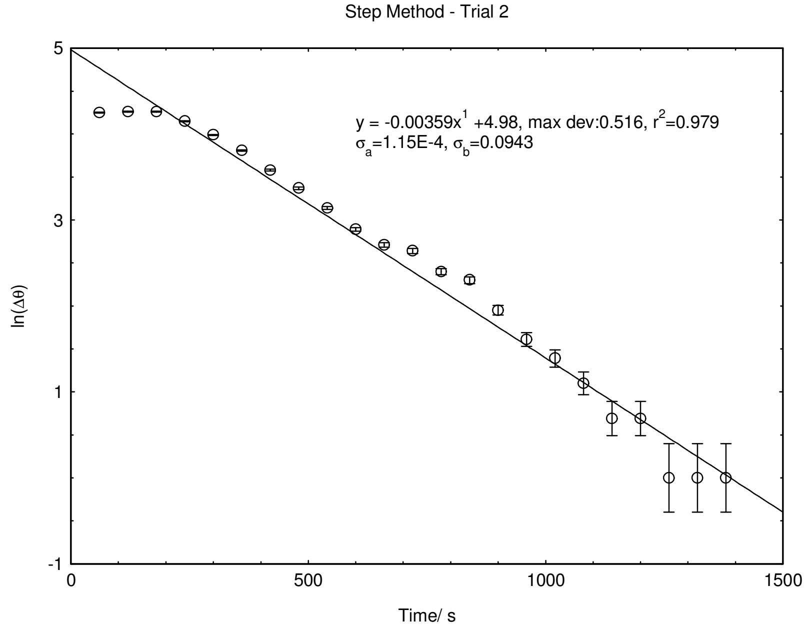

Figure 3 – Results of Trial 2 of the Step Method

From Equation 5 we find that: $D = 5.60 \times 10^{-8}\, m^2 s^{-1}$

Using the standard methods[3] the value of Chi Squared is found to be; $\chi^2 = 3187$. However this value of Chi squared is too large, it can be decreased to a more acceptable value by discounting the first five and last six points from the experiment.

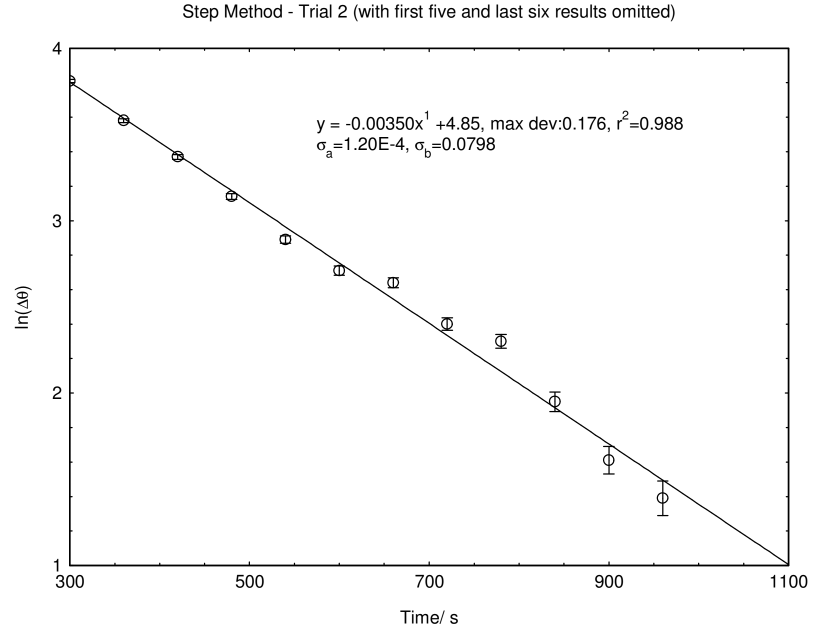

Figure 4 – Results of Trial 2 of the Step Method (with first five and last six points removed)

Removing these points gives a more acceptable value of 54.74 for Chi Squared.
This gives: $m = -0.0035 s^{-1} \Rightarrow D = 5.50 \times 10^{-8}\, m^2 s^{-1}$ (By Equation 5)

4.2. Results from the Periodic Method – Amplitude

4.2.1. Determining the axial temperature

For the periodic method the plot oscillated between the maximum and minimum axial temperatures. At the end of the experiment (after a few periods had been completed), the axial temperature was allowed to reach 100 ℃ and then 0 ℃. This was done so that the maximum and minimum temperatures could be determined. The equivalent temperature for each of the smallest divisions is marked on each plot.

4.2.2. Trial 1

For the first trial a time period of 360s (i.e. leaving the resin in the ice water for 3 minutes then in the boiling water for 3 minutes, then repeating)

The following was recorded:

$\theta_1 = 0^\circ C \qquad \theta_2 = 100^\circ C$

$B = 74.2 - 30.9 = 43.3^\circ C$

$T = 360s$

$\Rightarrow M_0(x) = 2.94$ (By Equation 7a)

To determine the value of the diffusivity, a plot of x against the Kelvin function was used;

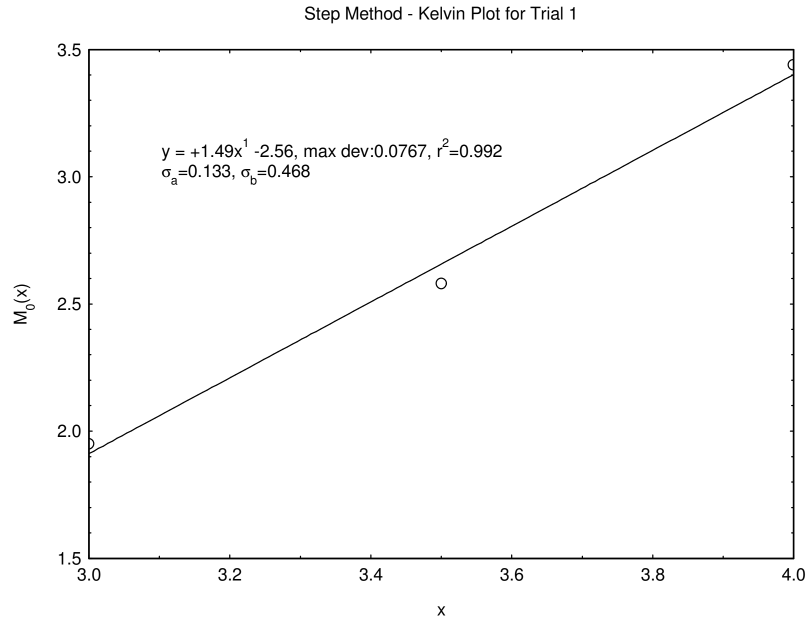

Figure 5 – Kelvin plot for trial 1 of the Periodic Method - Amplitude

From this plot we find that $x = 3.69 = a\sqrt{\dfrac{2\pi}{TD}}$ (By Equation 8)

$$\Rightarrow D = 11.60 \times 10^{-8} m^2 s^{-1}$$

4.2.3. Trial 2

For this second trial a time period of 120s (i.e. leaving the resin in the ice water for 1 minute then in the boiling water for 1 minute, then repeating)

The following was recorded:

$\theta_1 = 0^\circ C$

$\theta_2 = 100^\circ C$

$B = 26.8^\circ C$

$T = 120s$

$\Rightarrow M_0(x) = 4.75$ (By Equation 7a)

As before a plot of the Kelvin function is required:

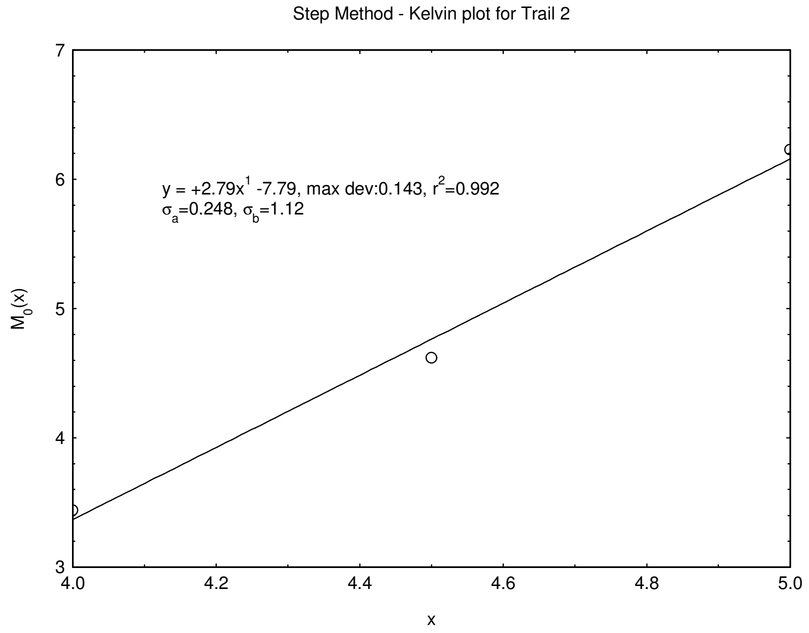

Figure 6 – Kelvin plot for trial 2 of the Periodic Method - Amplitude

From this we find that $x = 4.49 = a\sqrt{\frac{2\pi}{TD}}$ (By Equation 8)

$$\Rightarrow D = 23.60 \times 10^{-8} m^2 s^{-1}$$

4.3. Results from the Periodic Method – Phase Difference

4.3.1. Trial 1

As for the first trial of the Amplitude method, a time period of 360s was used first;

The following was recorded:
$T = 360s$
$\Delta t = 60s$

$\phi = \frac{\pi}{3} = 1.047 = \arg[M_0(x)]$ (By Equation 7b)

To determine the value of the diffusivity, a plot of x against the argument of the Kelvin function was used;

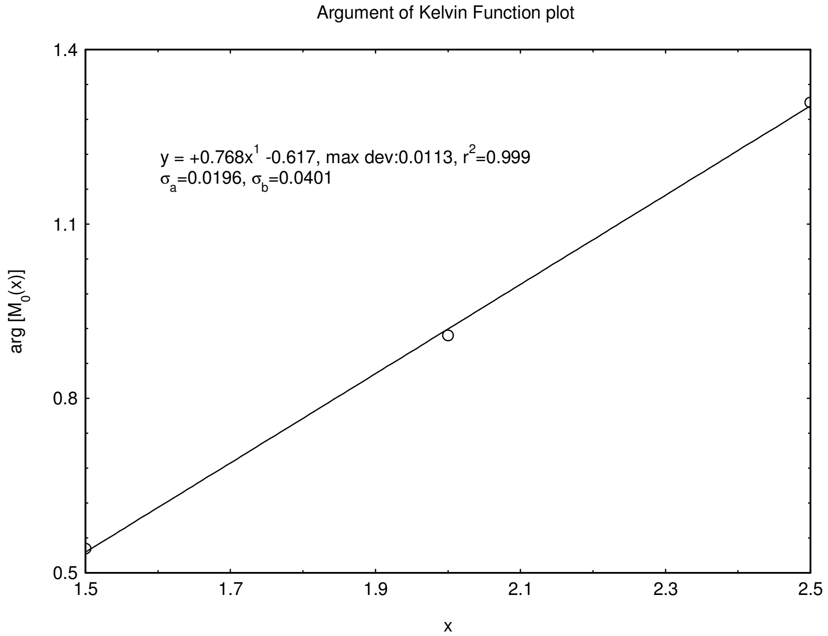

Figure 7 – Argument of Kelvin plot for trial 1 of the Periodic Method - Phase

From this plot we find that $x = 2.17 = a\sqrt{\frac{2\pi}{TD}}$ (By Equation 8)

$$\Rightarrow D = 33.80 \times 10^{-8} m^2 s^{-1}$$

4.3.2. Trial 2

Again for this second trial a time period of 120s was used

The following was recorded:

The following was recorded:

$T = 120s$

$\Delta t = 60s$

$\phi = \pi = 3.142 = \arg[M_0(x)]$ (By Equation 7b)

As before a plot of the Kelvin function is used:

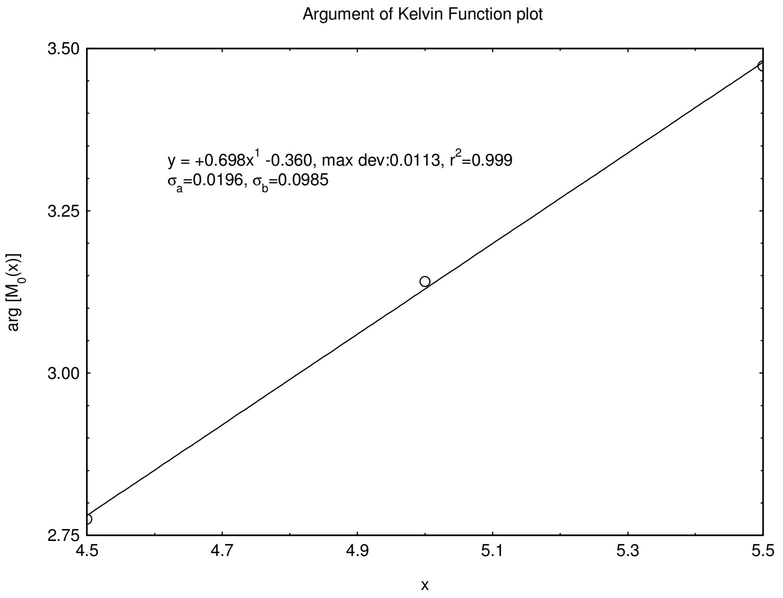

Figure 8 – Argument of Kelvin plot for trial 2 of the Periodic Method - Phase

From this we find that $x = 5.02 = a\sqrt{\frac{2\pi}{TD}}$ (By Equation 8)

$$\Rightarrow D = 18.95 \times 10^{-8} m^2 s^{-1}$$

4.4. Errors

There are two types of errors[3];

- **Systematic errors**; which are flaws in the experiment. The systematic errors of this experiment include:

  1. The assumption that the resin would be cooled to 0 ºC or heated to 100 ºC, for this to actually happen the temperature of the ice water would need to be less than 0 ºC and the temperature of the boiling water would need to be greater than 100.ºC

  2. The assumption that the heat density in the ice water and the boiling water were uniform. As they were not, the axial temperature would vary along the axis of the resin.

  The way to eliminate these systematic errors would be to use more accurate equipment and to use different mediums to cool and to heat the resin. For example a lower range of temperatures could be achieved if liquid nitrogen were used in place of the ice water.

- **Random errors;** which are imperfect readings due to the error of the equipment we can use and the human error involved. These can e accounted for. From our base measurements we evaluated the errors to be:

  - Error on diameter (as given by Vernier Callipers): $\pm 0.05mm$
  - Error on a temperature reading ± 0.5 *small squares* this is different for the two trials as the square equate to differing temperatures.

4.4.1. Calculation of errors for the Step Method

Using the standard methods[3] the error for the value of the diffusivity as given by the Step Method is given by;

$$D = -\frac{md^2}{4\lambda_1^2} \qquad \text{[Equation 5]}$$

$$\left(\frac{\sigma_D}{D}\right)^2 = \left(\frac{\sigma_m}{m}\right)^2 + 4\left(\frac{\sigma_d}{d}\right)^2$$

$$\Rightarrow \sigma_D = D\sqrt{\left(\frac{\sigma_m}{m}\right)^2 + 4\left(\frac{\sigma_d}{d}\right)^2} \qquad \text{[Equation 9]}$$

For Trial 1; $\sigma_D = 0.57 \times 10^{-8} m^2 s^{-1}$
For Trial 2: $\sigma_D = 0.19 \times 10^{-8} m^2 s^{-1}$

4.4.2. Calculation of errors for the Periodic Method - Amplitude

The error for the value of the diffusivity as given by the Periodic Method is given by[3];

$$M_0(x) = \frac{4}{\pi}\frac{\theta_2 - \theta_1}{B} \qquad \text{[Equation 7a]}$$

$$\left(\frac{\sigma_{M_0}}{M_0}\right)^2 = \left(\frac{\sigma_{\theta_2-\theta_1}}{\theta_2 - \theta_1}\right)^2 + 4\left(\frac{\sigma_B}{B}\right)^2$$

$$\Rightarrow \sigma_{M_0} = M_0\sqrt{\left(\frac{\sigma_{\theta_2-\theta_1}}{\theta_2 - \theta_1}\right)^2 + 4\left(\frac{\sigma_B}{B}\right)^2} \qquad \text{[Equation 10]}$$

This gives that $\sigma_{M_0} = 0.052$ for Trial 1 and $\sigma_{M_0} = 0.13$ for Trail 2.

This error in the value given by the Kelvin function can now be calculated using the Kelvin plots (see Figures 5 and 6) with the outcome being the error in the value of x

For Trail 1:

$$x\left(M_0 + \sigma_{M_0}\right) = 3.72 \qquad \Rightarrow \sigma_{x_1} = 3.72 - 3.69 = 0.03$$
$$x\left(M_0 - \sigma_{M_0}\right) = 3.65 \qquad \Rightarrow \sigma_{x_2} = 3.69 - 3.65 = 0.04$$

Hence we can find a mean error; $\sigma_x = \frac{\sigma_{x_1} + \sigma_{x_2}}{2} = \frac{0.03 + 0.04}{2} = 0.035$

For Trail 2:

$$x\left(M_0 + \sigma_{M_0}\right) = 4.54 \qquad \Rightarrow \sigma_{x_1} = 4.54 - 4.49 = 0.05$$
$$x\left(M_0 - \sigma_{M_0}\right) = 4.45 \qquad \Rightarrow \sigma_{x_2} = 4.49 - 4.45 = 0.04$$

Again taking a mean error; $\sigma_x = \frac{\sigma_{x_1} + \sigma_{x_2}}{2} = \frac{0.05 + 0.04}{2} = 0.045$

As x is a unitless quantity, so too is the error on x.

By substituting and rearranging Equation 8;

$$\Rightarrow D = \frac{\pi}{2T}\left(\frac{d}{x}\right)^2 \qquad \text{[Equation 11]}$$

It is found that[3]; $\left(\frac{\sigma_D}{D}\right)^2 = 4\left(\frac{\sigma_d}{d}\right)^2 + 4\left(\frac{\sigma_x}{x}\right)^2$

This rearranges to; $\sigma_D = 2D\sqrt{\left(\frac{\sigma_d}{d}\right)^2 + \left(\frac{\sigma_x}{x}\right)^2}$ [Equation 12]

For Trial 1; $\sigma_D = 0.20\times10^{-8} m^2 s^{-1}$
For Trial 2: $\sigma_D = 0.50\times10^{-8} m^2 s^{-1}$

4.4.3. Calculation of errors for the Periodic Method – Phase Difference

The error for the value of the diffusivity as given by the Periodic Method is given by[3];

$$\phi = \frac{2\pi\,\Delta t}{T} = \arg[M_0(x)] \qquad \text{[Equation 7b]}$$

$$\left(\frac{\sigma_{\arg[M_0]}}{\arg[M_0]}\right)^2 = \left(\frac{\sigma_{\Delta t}}{\Delta t}\right)^2 + \left(\frac{\sigma_T}{T}\right)^2$$

$$\sigma_{\arg[M_0]} = \arg[M_0]\sqrt{\left(\frac{\sigma_{\Delta t}}{\Delta t}\right)^2 + \left(\frac{\sigma_T}{T}\right)^2} \qquad \text{[Equation 13]}$$

Given the following for both trials;
$\sigma_{\Delta t} = \pm 30s$
$\sigma_T \pm 0.5s$

This gives that $\sigma_{\arg[M_0]} = 0.524$ for Trial 1 and $\sigma_{\arg[M_0]} = 1.571$ for Trail 2.

This error in the value given by the argument of the Kelvin function can now be calculated using the argument Kelvin plots (see Figures 7 and 8) with the outcome being the error in the value of x

For Trail 1:

$$x(\arg[M_0] + 0.524) = 2.849 \Rightarrow \sigma_{x_1} = 0.679$$
$$x(\arg[M_0] - 0.524) = 2.035 \Rightarrow \sigma_{x_2} = 0.135$$

Hence we can find a mean error; $\sigma_x = \frac{\sigma_{x_1} + \sigma_{x_2}}{2} = \frac{0.679 + 0.135}{2} = 0.407$

For Trail 2:

$$x(\arg[M_0]+1.571)=7.268 \Rightarrow \sigma_{x_1}=4.126$$
$$x(\arg[M_0]-1.571)=2.766 \quad \Rightarrow \sigma_{x_2}=0.376$$

Again taking a mean error; $\sigma_x = \frac{\sigma_{x_1}+\sigma_{x_2}}{2} = \frac{4.126+0.376}{2} = 2.251$

As x is a unitless quantity, so too is the error on x.

As for the Amplitude method; $\sigma_D = 2D\sqrt{\left(\frac{\sigma_d}{d}\right)^2 + \left(\frac{\sigma_x}{x}\right)^2}$ [Equation 12]

For Trial 1; $\sigma_D = 1.10.\times 10^{-8} m^2 s^{-1}$
For Trial 2: $\sigma_D = 0.35 \times 10^{-8} m^2 s^{-1}$

4.5. Final Answer

From the four trials (two in each method), it was found that;

Step Method – Trial 1	$D=(13.70\pm 0.57)\times 10^{-8} m^2 s^{-1}$
Step Method – Trial 2	$D=(5.50\pm 0.19)\times 10^{-8} m^2 s^{-1}$
Periodic Method – Amplitude Trial 1	$D=(11.60\pm 0.23)\times 10^{-8} m^2 s^{-1}$
Periodic Method – Amplitude Trial 2	$D=(23.60\pm 0.49)\times 10^{-8} m^2 s^{-1}$
Periodic Method –Phase Trial 1	$\sigma_D=(33.8\pm 1.10).\times 10^{-8} m^2 s^{-1}$
Periodic Method – Phase Trial 2	$\sigma_D=(18.95\pm 0.35)\times 10^{-8} m^2 s^{-1}$

Using these results a weighted mean[3] can be used to arrive at a final answer for the diffusivity;

The weighted mean of the diffusivity: $\overline{D} = 7.86\times 10^{-8} m^2 s^{-1}$

The error associated with this weighted mean; $\sigma_{\overline{D}} = 0.22\times 10^{-8} m^2 s^{-1}$

Hence the final answer is $\overline{D} = (9.45\pm 0.22)\times 10^{-8} m^2 s^{-1}$

5. Conclusion

From the experiment it was found that the value of the diffusivity of our cylindrical piece of resin is $D = (9.45 \pm 0.22) \times 10^{-8} m^2 s^{-1}$

A linear relationship was demonstrated, the graph of which had a negative gradient, this showed agreement with Equation 2.

In the future, the errors could be reduced to show this relationship more clearly in a number of ways. Firstly we could repeat the same method, although this would require much time to return any significant improvement in the value of the diffusivity.

The method could be modified such that the measurements are more accurate. For example a greater range of temperatures could be achieved by making use of different cooling methods.

Alternatively the diffusivity of the sample could be calculated if the thermal conductivity, the density and the specific heat of the resin could be found using the relationship[3];

$$D = \frac{K}{\rho C}$$

6. References

1. **G. Stephenson**, *An Introduction to Partial Differential Equations for Science Students,* **Longman, Essex, 1988**

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4. Charles Kittel & Herbert Kroemer, *Thermal Physics 2nd Edition*, **W. H. Freeman, 1980**